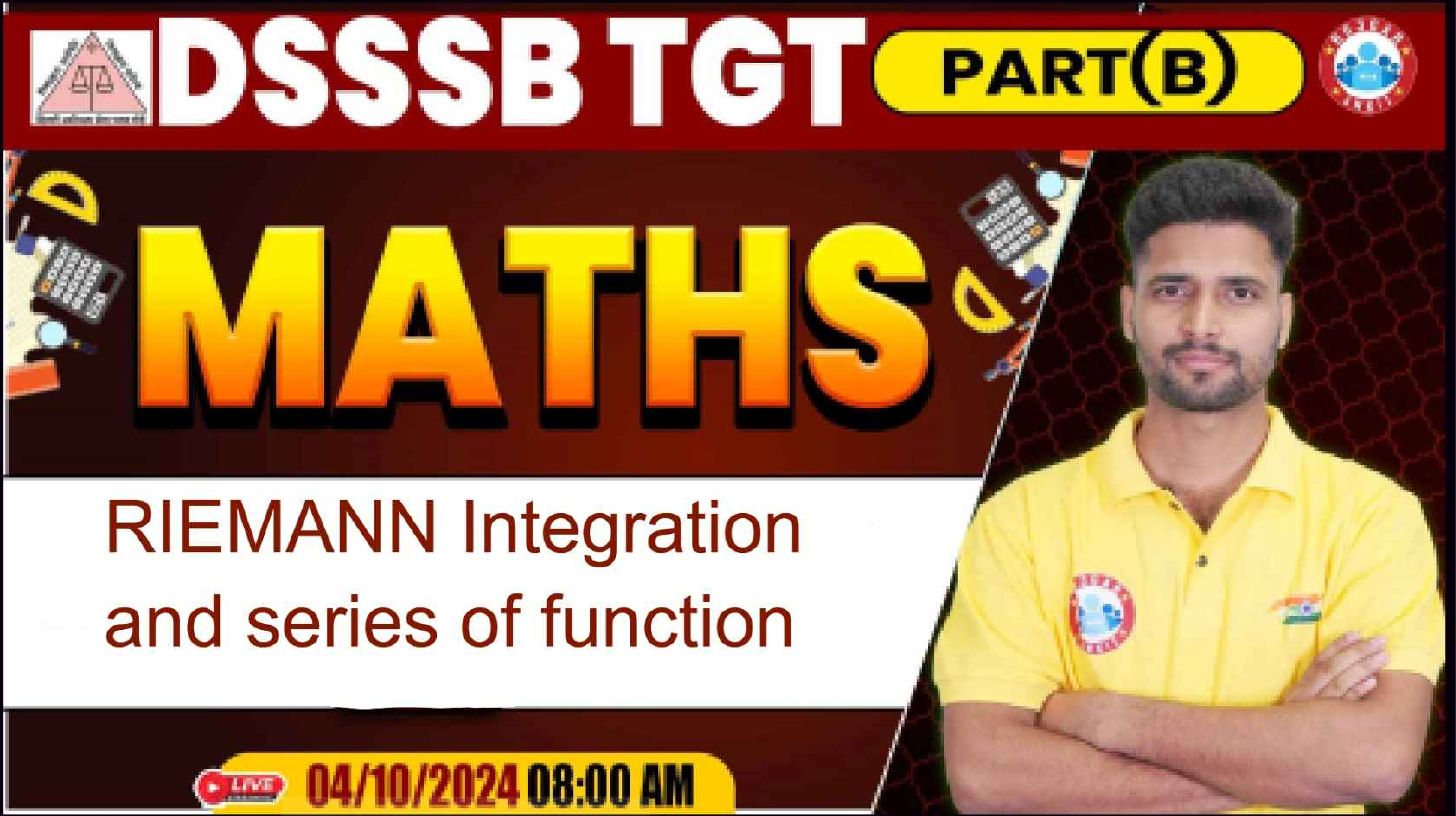

DSSSB TGT
PART(B)
MATHS
RIEMANN Integration
and series of function
LIVE
04/10/2024 08:00 AM

चरघातांकी

Exponential Function And Exponential Series

Let x be a real number and e be the base of a natural logarithm, then the series expansion of exponential function of x i.e e^x can be expressed as,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$$

exponetial funⁿ

Expontial series.

Note:-

1 Here value of $e \cong 2.718$

2. $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \infty$

Exponential Function of Complex Number

Let $z = x + iy$ be a complex number, then the series expansion of exponential function of complex number Z i.e e^z can be written as,

$$e^Z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \infty$$

And the Exponential function e^z of the complex number $z = x + iy$, where x and y are real numbers, is defined as

$$e^Z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos\ y + i\sin\ y)$$

i.e $e^{x+iy} = e^x(\cos y + i\sin y)$

Note:-

By Euler's Theorem, $e^{iy} = \cos y + i\sin y$

Circular Functions of Complex Numbers

The basic fact is that, $\sin x, \cos x$ and $\tan x$ etc. are trigonometric functions only when x is a real quantity. In the case when x is replaced by a complex quantity, z then they are known as circular functions.

Polar's ↔ Euler's

↓ ↓

Sin/Cos Sin/Cos

✓ ✓

Definition:-

For all real values of x, we know that

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

(+)

Adding and subtracting (1) and (2), we get

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} \text{ and } \sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

$$e^{ix} = \frac{e^{ix}+e^{-ix}}{2} + i\cdot\frac{e^{ix}-e^{-ix}}{2i}$$

$$e^{ix} \Rightarrow \frac{2e^{ix}}{2}$$

Which are called Euler's exponential values of $\sin x$ and $\cos x$.

If $z = x + iy$ be a complex number, then circular functions $\cos z$ and $\sin z$ are defined as follows

$\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin Z = \frac{e^{iz}-e^{-iz}}{2i}$

Now by above these two circular functions, we can also defined other circular functions.

$$\tan Z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

$$\cot z = \frac{\cos z}{\sin Z} = \frac{i(e^{iz} + e^{-iz})}{(e^{iz} - e^{-iz})}$$

$$\sec Z = \frac{1}{\cos Z} = \frac{2}{e^{iz} + e^{-iz}}$$

$$\operatorname{cosec} Z = \frac{1}{\sin Z} = \frac{2i}{e^{iz} - e^{-iz}}$$

Expansion of $\cos z$ and $\sin z$

We know that

$$e^{Z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \infty$$

Now replace z by iz, then we get

$$e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \cdots \infty$$

$$e^{iz} = 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \infty$$

$$e^{iZ} = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \infty\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \infty\right) \ldots\ldots$$

$$e^{-iZ} = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \infty\right) - i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \infty\right) \cdots$$

Adding (i) and (ii), we get ⊕ (Even power)

$$e^{iZ} + e^{-iZ} = 2\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \infty\right)$$

OR

$$\frac{e^{iZ} + e^{-iZ}}{2} = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \infty\right)$$

Hence $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \infty$

Again subtracting (ii) from (i), we get

⊖ (odd power)

$$e^{iZ} - e^{-iZ} = 2i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \infty\right)$$

OR

$$\frac{e^{iZ} - e^{-iZ}}{2i} = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \infty\right)$$

Hence

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \infty$$

Note:-

$$\tan z = z + \frac{z^3}{3} + \frac{2}{15} z^5 + \cdots \infty$$

De Moivre's Theorem

If θ be complex and n be any integer, positive or negative, then by Euler's Theroem, we have

$$\boxed{\cos\,\theta + i\sin\,\theta = e^{i\theta}} \rightarrow \text{Euler's form}$$

$$\Rightarrow (\cos\,\theta + i\sin\,\theta)^n = e^{in\theta}$$

$$= \cos\,n\theta + i\sin\,n\theta$$

Hence $\boxed{(\cos\,\theta + i\sin\,\theta)^n = \cos\,n\theta + i\sin\,n\theta}$ → De-moivre's theorem.

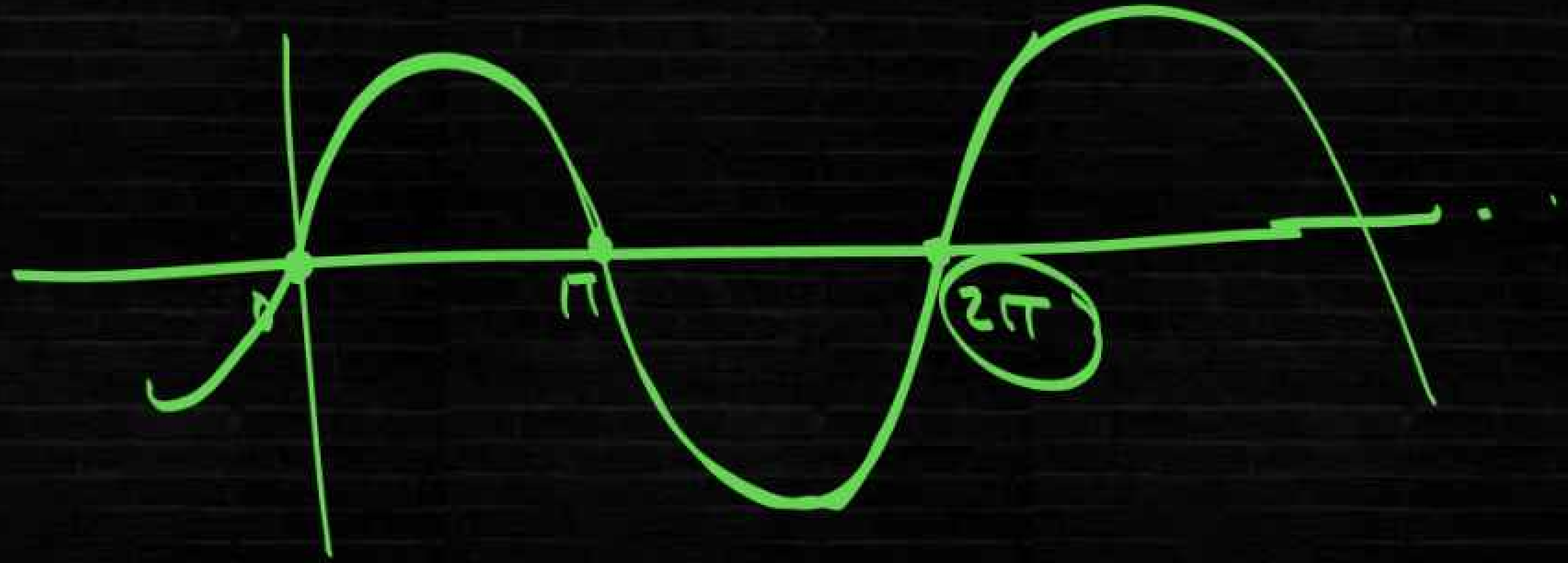
This result is true for all values of θ, real or complex.

Periodicity of Circular Functions

1 $\sin z$ and $\cos z$ are periodic functions with period 2π.

2 $\text{cosec } z$ and $\sec z$ are also periodic functions with period 2π.

3 $\tan z$ and $\cot z$ are periodic functions with period π.



Hyperbolic Functions

Let θ be a real or complex number, then we define hyperbolic sine and cosine as follows

$$\sinh\ \theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

which is read as hyperbolic sine θ

And $\cosh\ \theta = \frac{e^{\theta}+e^{-\theta}}{2}$

known as hyperbolic cosine θ

Similarly, we can define other hyperbolic functions as follows:-

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta} = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$$

$$\coth\theta = \frac{\cosh\theta}{\sinh\theta} = \frac{e^{\theta} + e^{-\theta}}{e^{\theta} - e^{-\theta}}$$

$$\text{cosech}\ \theta = \frac{1}{\sinh\theta} = \frac{2}{e^{\theta} - e^{-\theta}}$$

$$\text{sech}\ \theta = \frac{1}{\cosh\theta} = \frac{2}{e^{\theta} + e^{-\theta}}$$

Note:-

1 $\cosh\theta + \sinh\theta = e^{\theta}$

2 $\cosh\theta - \sinh\theta = e^{-\theta}$

3 $\sinh 0 = 0, \cosh 0 = 1, \tanh 0 = 0$

Expansions of $\sinh\theta$ and $\cosh\theta$ in powers of θ

As we know that,

$$\sinh\theta = \frac{1}{2}\left[e^{\theta} - e^{-\theta}\right]$$

$$= \frac{1}{2}\left[\left(1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots\infty\right) - \left(1 - \theta + \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \cdots\infty\right)\right]$$

$$\sinh\theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\infty$$

Similarly,

$$\cosh\theta = \frac{1}{2}\left[e^{\theta} + e^{-\theta}\right]$$

$$= \frac{1}{2}\left[\left(1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots\infty\right) + \left(1 - \theta + \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \cdots\infty\right)\right]$$

$$\cosh\theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots\infty$$

Periodicity of Hyperbolic Functions

As we know that,

$$\sinh\ \theta = \frac{1}{2}\left[e^{\theta} - e^{-\theta}\right]$$

$$\therefore \sinh\ (\theta + 2n\pi i) = \frac{1}{2}\left[e^{\theta+2n\pi i} - e^{-(\theta+2n\pi i)}\right], n \in Z$$

$$= \frac{1}{2}\left[e^{\theta} \cdot e^{2n\pi i} - e^{-\theta} \cdot e^{-2n\pi i}\right]$$

$$= \frac{1}{2}\left[e^{\theta} - e^{-\theta}\right] = \sinh\ \theta$$

$$\begin{bmatrix} \because e^{2n\pi i} = \cos\ 2n\pi + i\sin\ 2n\pi = 1 \\ e^{-2n\pi i} = \cos\ 2n\pi - i\sin\ 2n\pi = 1 \end{bmatrix}$$

Thus, $\sinh\ \theta$ remains unchanged when θ is increased by any multiple of $2\pi i$.

Hence $\sinh\ z$ is a periodic function with period $2\pi i$.

Similarly, we can say that

1. $\cosh\ z$ is a periodic function with period $2\pi i$.
2. $\text{cosech}\ z$ and $\text{sech}\ z$ are also periodic functions with period $2\pi i$.
3. $\tanh\ z$ and $\coth\ z$ are periodic functions with period πi.

Relation between Circular Functions and Hyperbolic Functions

We know that

$$\sin\ \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\therefore \sin(ix) = \frac{e^{i^2x} - e^{-i^2x}}{2i} \text{ [Put } \theta = ix]$$

$$= \frac{e^{-x} - e^{x}}{2i}$$

$$= -i\frac{e^{x} - e^{-x}}{2i^2} = i\frac{e^{x} - e^{-x}}{2} = i\sinh\ x$$

Thus $\sin(ix) = i\sinh x$

Similarly,

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\therefore \cos(ix) = \frac{e^{i^2x}+e^{-i^2x}}{2} \text{ [Put } \theta = ix]$$

$$= \frac{e^{-x}+e^{x}}{2}$$

$$= \frac{e^{x}+e^{-x}}{2} = \cosh x$$

Thus $\cos(ix) = \cosh x$

Simlarly,

$$\tan(ix) = \frac{\sin(ix)}{\cos(ix)} = \frac{i\sinh x}{\cosh x} = i\tanh x$$

$$\cot(ix) = \frac{\cos(ix)}{\sin(ix)} = \frac{\cosh x}{i\sinh x} = -i\coth x$$

$$\sec(ix) = \frac{1}{\cos(ix)} = \frac{1}{\cosh x} = \text{sech } x$$

$$\text{cosec}(ix) = \frac{1}{\sin(ix)} = \frac{1}{i\sinh x} = -i\text{cosech } x$$

Note:-

1. $\sinh(ix) = i\sin x$
2. $\cosh(ix) = \cos x$
3. $\tanh(ix) = i\tan x$